Composite-Fermion Approach for the Fractional Quantum Hall Effect

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In the standard hierarchical scheme the daughter state at each step results from the fractional quantum Hall effect of the quasiparticles of the parent state. In this paper a new possible approach for understanding the fractional quantum Hall effect is presented. It is proposed that the fractional quantum Hall effect of electrons can be physically understood as a manifestation of the integer quantum Hall effect of composite fermionic objects consisting of electrons bound to an even number of flux quanta.

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Even though the experimental observations of the integer[1] and the fractional[2] quantum Hall effect[3] (QHE) are essentially identical, except for the value of the quantized Hall resistance, there are, roughly speaking, three different theoretical schemes for their explanation. While the integer QHE (IQHE) is thought of essentially as a noninteracting electron phenomenon,[4] the fractional QHE (FQHE) is believed to arise from a condensation of the two-dimensional (2D) electrons into a "new collective state of matter"[5] as a result of interelectron interactions. Even within the FQHE the "fundamental" fractions $\frac{1}{3}, \frac{1}{5}, \ldots$ play a special role and the other fractions are obtained in a hierarchical scheme[6] in which a daughter state is obtained at each step from a condensation of the quasiparticles of the parent state into a correlated low-energy state.

The purpose of this Letter is to present a theoretical framework which enables an understanding of both the IQHE and the FQHE in a unified scheme as two different manifestations of the same underlying physics. It is argued that the possibility of QHE at fractional filling factors $p/(2mp\pm 1)$, where m and p are integers, arises because the correlations in the phase factors at these filling factors are very similar to the correlations present at integer filling factors p. This approach not only gives all the observed fractions (except[7] $\frac{5}{2}$, which therefore requires some additional physics[8]), and explains in doing so why only fractions with odd denominators are observed, but also provides the order of their stability, in agreement with experiments. Furthermore, it suggests a generalization of the Laughlin wave functions to other fractions.

I start by proposing a remarkably simple picture for understanding the origin of the FQHE. The important parameter is the ratio of the total number of flux quanta ($\phi_0 = hc/e$) to the total number of electrons, which is the inverse of the filling factor ν (in the thermodynamic limit) and specifies the average number of flux quanta available to each electron. Consider a 2D electron gas in the presence of a transverse magnetic field at an integer filling factor $\nu = p$, so that there is an average flux ϕ_0/p per electron. The electronic wave function $\Psi_{\pm p}$ ($\pm$ corresponds to magnetic field in the $\mp z$ direction) in this situation is rather insensitive to the details of the interelectron interactions and is determined mainly by virtue of the fermionicity of the electrons. Thus, the long-range correlations due to the Fermi statistics provide rigidity to the electron system at integer filling factors which results in the phenomenon of IQHE. It is useful to think in the path-integral language:[9] The partition function gets contributions from the closed paths in the configuration space (for example, a path in which one electron moves in a loop while the others are held fixed, or a cooperative ring exchange path[9]). The phase associated with each closed path has two contributions: the Aharonov-Bohm phase which depends on the flux enclosed in the loop, and the statistical phase which depends on how many electrons participate in the path. An incompressible state is obtained at integer filling factors presumably because of some special correlations (which may not be easily identified) built in the phase factors corresponding to the various paths. Now attach to each electron an infinitely thin magnetic solenoid carrying a flux $\alpha\phi_0$ (pointed in the $-z$ direction). For lack of a better name, we term an electron bound to a flux tube a "composite particle." As is well known,[10] the statistics of the composite particles is in general fractional, and is such that an exchange of two composite particles produces a phase factor $(-1)^{1+\alpha}$ (Ref. 11). The relevant case here is when α is equal to an even integer ($\alpha = 2m$), and the composite particles are fermions. It is easy to see that in this case the phase factor acquired along a given closed path is identical to the phase factor acquired in the absence of the flux tubes, implying that the correlations in the phase factors for $\alpha = 2m$ are the same as those for $\alpha = 0$. Since these correlations are responsible for rigidity and QHE at integer filling factors, one can expect the composite fermion state $\Psi^{2m}_{\pm p}$, which is obtained by adding to each electron in $\Psi_{\pm p}$ a flux $2m\phi_0$, to also be rigid and show QHE.

To determine the filling factor of $\Psi^{2m}_{\pm p}$ we exploit an ingenious observation due to Arovas *et al.*[10] and Laughlin:[11] A (uniform) liquid of electrons, each carrying with it a flux $\alpha\psi_0$, is equivalent, *in a mean field sense* to a (uniform) liquid of electrons in a magnetic field of strength such that there is an average flux of $\alpha\phi_0$ per electron. A uniform electron density is required to produce a uniform flux density. Since in the state $\Psi^{2m}_{\pm p}$ there are a total of $2m \pm p^{-1}$ flux quanta per electron, we identify it with the mean-field state of *electrons* at fractional filling $\nu = p/(2mp \pm 1)$. It must be borne in mind that the true electron state is not as rigid as the composite fermion state, because in the true state the flux tubes are not strictly bound to the electrons, and the phase factors simulate IQHE only on average. However, *provided that the true electron state is also incompressible*, the composite fermion state should provide the correct description of the essential physics at the mean-field level, as it contains the correlations giving rise to the incompressibility. On the other hand, when the true electron state is not incompressible, identification of the composite fermion state with the mean-field state of the electrons is no longer valid, or meaningful.

Thus in this approach there are two types of correlations essential for FQHE. The first type of correlations, which have been widely appreciated in the field,[12,13] involve binding of electrons and zeros of the wave function, or, equivalently, of electrons and flux tubes, which is a very useful way of incorporating the effect of repulsive interactions. Thus the role of repulsive interactions in the present framework is assumed to be to generate composite fermions. The second type of correlations, that impart rigidity to the composite fermion system and thus lead to FQHE, are the correlations due to their Fermi statistics. These are included in the present scheme by mimicking the statistical correlations present in the noninteracting electron system at integer filling factors. This is the central idea of this work; it is best summarized by saying that *the FQHE of electrons is a manifestation of the IQHE of composite fermions.*

The Hall plateaus at fractional filling factors appear in this model precisely as at integer filling factors except for the trivial modification that now each electron carries with it $2m$ flux quanta. Following the argument of Laughlin and Halperin[4] consider a corbino disk geometry. The Hall resistance is related to the charge transported from one edge to the other as one flux quantum is adiabatically pierced through the center. At integer filling factor p, p electrons are transported across the sample in this process. For $\Psi^{2m}_{\pm p}$, as each electron carries $2m$ flux quanta, one must supply $2mp$ additional flux quanta (in all $2mp \pm 1$ flux quanta) to transport p electrons across the sample. This gives $R_H = h/\nu e^2$ with $\nu = p/(2mp \pm 1)$. Just as in the IQHE, sample impurities and inhomogeneities create localized states, which produce a quantized Hall plateau so long as the Fermi level lies in a mobility gap.[4]

The stable fractional filling factors obtained in this manner are $p/(2mp \pm 1)$, and due to electron-hole symmetry, $1 - p/(2mp \pm 1)$. [As indicated by Haldane[8,14] this implies possible stability at fractions $n + p/(2mp \pm 1)$ and $n + 1 - p/(2mp \pm 1)$ in the nth Landau level (LL).] Notice that only fractions with odd denominators appear. In fact, in the present framework QHE at fractional values of ν with odd denominators is as natural as the QHE at integer values of ν. Besides explaining the "odd denominator rule," we are also able to predict the order of stability of the fractions, or the order in which new fractions should appear as the sample quality is improved. Since a collapse of the gap due to an "unbinding transition" is more likely for larger values of m, if a fraction $p/(2mp \pm 1)$ is observed for a given p then all fractions $p/(2m'p \pm 1)$ with $m' < m$ must also be observed. One also expects weaker correlations for higher values of p. Thus in Fig. 1 a given fraction in the right (left) half is more stable than the one directly above it and the one on its right (left). This is quite generally borne out in experiments.[7,15–17] This also identifies the fractions to be observed next, if any, as the sample quality is further improved. Read[18] has pointed out that the fractions obtained here are only the first level of a *new* hierarchy, and all other fractions with odd denominators can be obtained within the present formalism. However, it is interesting to note that all the observed fractions, except $\frac{4}{11}$ and $\frac{4}{13}$ (Ref. 19), are obtained in this scheme at the very first level, which is to be contrasted with the standard hierarchy in which one needs to go down many

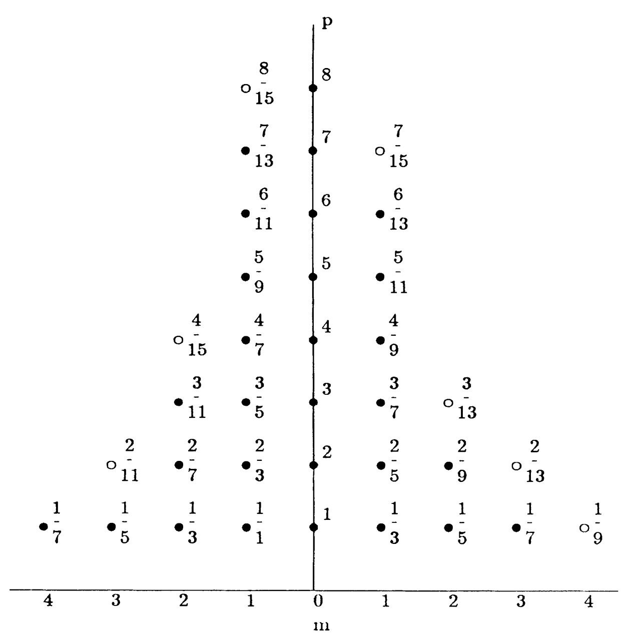

FIG. 1. The fractions $p/(2mp+1)$ and $p/(2mp-1)$ are shown in the right half and the left half, respectively. The filled circles show the fractions that have been observed in the lowest LL. The predicted values of the next most stable fractions at this level are shown near empty circles.

levels in order to obtain some of the observed fractions. It is also worth mentioning that the present scheme naturally produces the experimentally observed sequences[7,15] of fractions converging to $\frac{1}{2}$ (for $m=1$), to $\frac{1}{4}$ (for $m=2$), to $\frac{3}{4}$ (hole analog of $\frac{1}{4}$), etc.

In the following I will construct explicit trial wave functions, analogous to the Laughlin wave functions, which have the correlations discussed above. The Hamiltonian for N noninteracting electrons ($N\rightarrow\infty$) at filling factor p is given by $H_0=\sum_{j=1}^{N}(2m_e)^{-1}(\mathbf{p}_j+e\mathbf{A}_j/c)^2$, where $\mathbf{A}_j$ is chosen so as to produce a uniform magnetic field in the $\mp z$ direction of strength such that there is an average flux of $p^{-1}\phi_0$ per electron. The corresponding ground-state wave functions are $\Psi_{\pm p}$ with $\Psi_{-p}=\Psi_{+p}^*$. We first consider the fractions $p/(2mp+1)$ which are obtained by starting from Ψ_{+p}. Gauge flux tubes $2m\phi_0$ are attached to each electron by adding to the vector potential $\mathbf{A}_j$ a singular gauge potential[10,11]

$$\mathcal{A}_j=-2m\frac{\phi_0}{2\pi}\sum_{k(k\neq j)}\nabla_j\theta_{jk},$$

where θ_{jk} is defined by $(z_j-z_k)=|z_j-z_k|\exp(i\theta_{jk})$, and $z_j=x_j+iy_j$ denotes the position (x_j,y_j) of the jth particle as a complex number. The new ground-state wave function is[11]

$$\Phi_{+p}^{2m}=\prod_{j<k}\frac{(z_j-z_k)^{2m}}{|z_j-z_k|^{2m}}\Psi_{+p}.$$

Clearly this is not an appropriate wave function for describing the FQHE of electrons. Following the analogy of the Laughlin wave functions, and for the reasons mentioned below, we write instead the following closely related (unnormalized) trial wave function

$$\Psi_{+p}^{2m}=\mathcal{Z}^{2m}\Psi_{+p},$$

where $\mathcal{Z}^{\alpha}=\prod_{j<k}(z_j-z_k)^{\alpha}$. This wave function has the same topological structure as Φ_{+p}^{2m}, and also describes electrons carrying gauge flux tubes of strength $2m\phi_0$. In this state addition of flux tubes is accompanied by a change in the size of the system in such a way as to keep the total flux per unit area (i.e., the magnetic field) constant.

This state has the following properties: (i) For $p=1$, Ψ_{+1}^{2m} is identical with the corresponding Laughlin state.[5] (ii) Since Ψ_{+p} is determined almost completely by the Pauli principle, and has little dependence on interelectron interactions, Ψ_{+p}^{2m} is also largely insensitive to the interactions. This is explicitly the case for the Laughlin states[5] which have been found to be very accurate for a variety of interelectron interactions. (iii) It describes an electron gas of uniform density. This follows straightforwardly from the fact that Ψ_{+p} describes an electron gas of uniform density. It is also an eigenstate of the angular momentum. (iv) One can read off the filling factor from the wave function. Take Ψ_{+p} with p LL's completely occupied in a disk-shaped region; the number of occupied single-particle states in each LL is N/p. Since the largest power of a z_j in $\mathcal{Z}^{2m}$ is $2m(N-1)$, Ψ_{+p}^{2m} has $2m(N-1)+Np^{-1}$ single-particle states occupied in each LL, which immediately yields a filling factor $p/(2mp+1)$ in the thermodynamic limit. Thus the state Ψ_{+p}^{2m} (unlike Φ_{+p}^{2m}) satisfies the fundamental requirement that the filling factor obtained by counting the total number of states agrees with that obtained from the flux-counting argument (i.e., the number of flux quanta piercing the sample is equal to the number of single-particle states in each LL). (v) The factor $\mathcal{Z}^{2m}$ in Ψ_{+p}^{2m} partially projects the single-particle states of the higher LL's into the lowest LL. Write

$$\Psi_{+p}^{2m}=A\mathcal{Z}^{2m}\prod_{j=1}^{N/p}\prod_{l=0}^{p-1}\zeta_{l,j-1}(z_{l+j}),$$

$$\zeta_{l,s}=(2\pi 2^{l+s}l!s!)^{-1/2}e^{|z|^2/4}\left(2\frac{\partial}{\partial z}\right)^{l}z^s e^{-|z|^2/2},$$

where A is the anitsymmetrization operator, $\zeta_{l,s}$ are the single-particle states, $l=0,\ldots,p-1$ is the LL index, and $s=0,\ldots,N/p-1$ is the angular momentum quantum number. $\mathcal{Z}^{2m}$ is a sum of terms of type $\prod_{j=1}^{N}z_j^{t_j}$ with $\sum_j t_j=mN(N-1)$, where t_j is *typically* a large power (in the thermodynamic limit) of order mN. Thus, in each term of Ψ_{+p}^{2m} the coordinate z_j of a particle appears as the product $z_j^{t_j}\zeta_{l,s}(z_j)$. For t_j of order mN, this product lies *almost entirely* in the lowest LL. Expanding it as a sum of single-particle states,

$$z_j^{t_j}\zeta_{l,s}=\sum_{k=0}^{l}a_k\zeta_{k,s+t_j+k-l},$$

one can show that the ratio a_{k+1}/a_k is of order $1/\sqrt{N}$; i.e., the amplitude of $z_j^{t_j}\zeta_{l,s}(z_j)$ is smaller by a factor of order $1/\sqrt{N}$ in each successively higher LL. Thus in Ψ_{+p}^{2m} the amplitude is expected to be in general much larger for the terms which have a greater number of the lowest LL states occupied. Furthermore, there are manifestly terms, with extremely large amplitudes, which have only the lowest LL occupied. This implies that, unless there are some very strange cancellations, the state Ψ_{+p}^{2m} lies predominantly in the lowest LL in the thermodynamic limit. (vi) Last, Ψ_{+p}^{2m} is expected to be a good variational state in the presence of repulsive interactions, because both the factors $\mathcal{Z}^{2m}$ and Ψ_{+p} are very efficient in keeping the electrons apart. This is a direct sense in which the correlations of the higher LL are utilized to obtain a low-energy state. Thus we believe that the states Ψ_{+p}^{2m} possess all the necessary properties of a reasonable trial state. At present we are working towards a quantitative test, which is complicated due to the complex structure and the inherent thermodynamic nature of these states. The form of the incompressible state at $\nu=p/(2mp-1)$, which is obtained starting from Ψ_{-p}, is not as obvious.

Normally the ground state is expected to be completely spin polarized, and is obtained by choosing in $\Psi_{\pm p}$ the p lowest LL's with the *same spin orientation*. However, when the spin splitting is insignificant, it may be useful to consider situations in which $\psi_{\pm p}$ has LL's with both spin orientations occupied, so that in $\Psi^{2m}_{\pm p}$ the two lowest spin-split Landau bands are occupied. Thus, for small spin splitting, there are in general many candidates for the incompressible state[12,14,20] for a given fraction: the completely spin polarized states, spin unpolarized states,[12,14] and partially spin polarized states.

There seems to be a close analogy between the composite fermion states proposed in this paper and the standard hierarchical states. To illustrate this, we consider the example of the $\frac{1}{3}$ state which is obtained by multiplying Ψ_{+1} by $\mathcal{Z}^2$. It can be shown[21] that the state Ψ_{+1} with one hole similarly produces the $\frac{1}{3}$ state with a Laughlin quasihole. By analogy, the $\frac{1}{3}$ state with a quasielectron would be obtained by multiplying by $\mathcal{Z}^2$ the state with the lowest ($l=0$) LL fully occupied and one electron in the $l=1$ LL. The state with fully occupied lowest LL and δ electrons in the $l=1$ LL then corresponds to the $\frac{1}{3}$ state with δ quasielectrons. Thus the $\frac{2}{5}$ state can be viewed in the present approach as the $\frac{1}{3}$ state with $N/2$ quasielectrons, and similarly the $\frac{3}{7}$ state can be viewed as the $\frac{2}{5}$ state with $N/3$ quasielectrons. This assignment is in exact agreement with that of the standard hierarchy theory.[6] One can also show that the quasiparticles described above have the same charge as those in the standard scheme.[21] The analogy is, however, not complete. In the standard picture stability is obtained when the quasielectrons form a Laughlin-type state, whereas in the composite-fermion scheme they derive their arrangement from the higher LL. Also, taking the above example, in the standard picture one can obtain both the $\frac{3}{7}$ and the $\frac{5}{13}$ states from the $\frac{2}{5}$ state, whereas the composite-fermion approach does not yield the experimentally unobserved $\frac{5}{13}$ state at this level.

In conclusion, this paper proposes that the FQHE can be accessed from the IQHE by adding an even number of flux quanta to each electron. This analogy between FQHE and IQHE suggests a natural generalization of the Laughlin states.

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